# On supermembrane actions on Calabi-Yau 3-folds

A. Imaanpur *
*Department of Physics, School of Sciences,*
*Tarbiat Modares University, P.O.Box 14155-4838, Tehran, Iran, and*
*Institute for Studies in Theoretical Physics and Mathematics*
*P.O.Box 19395-5531, Tehran, Iran*

August 16, 2000

**Abstract**

In this note we examine the supermembrane action on Calabi-Yau 3-folds. We write down the Dirac-Born-Infeld part of the action, and show that it is invariant under the rigid spacetime supersymmetry.

## 1 Introduction

The study of $p$-branes as solitonic solutions of supergravity has now become indispensable for the understanding of nonperturbative effects in superstring theory. On flat spacetimes $p$-branes are described by an action which consists of two parts; the Dirac-Born-Infeld (DBI) part, and the Wess-Zumino part. The latter can only be written down for some specific spacetime dimensions. For supermembranes, for instance, it exists only for 4, 5, 7, and 11 spacetime dimensions [1]. Both parts of the action are invariant under the rigid spacetime supersymmetry transformations, however, for a specific choice of normalization of the Wess-Zumino term, the whole action turns out to have an extra local symmetry known as $\kappa$-symmetry. The $\kappa$-symmetry allows one to remove the redundant fermionic degrees of freedom leaving only the physical ones.

$p$-branes (and also D-branes) have so far been studied on flat spacetimes, and in some cases on $AdS$ spaces (look at [1, 2] and the references therein, for instance). On Calabi-Yau manifolds, on the other hand, the action is not known, though, in some cases they have been studied through their low energy effective action [3, 4], which for D$p$-branes is just the dimensional reduction of 10d super Yang-Mills theory down to $p+1$ dimensions [5]. Along these lines, an effective action of D-branes on Calabi-Yau 3-folds was also introduced in [6].

The study of $p$-branes on Calabi-Yau 3-folds becomes crucial if we are to learn about the nonperturbative effects of superstring theory or M-theory compactified on such manifolds.

*Email: aimaanpu@theory.ipm.ac.ir

So it is of importance to find the effective action of $p$-branes on Calabi-Yau manifolds. This note is an attempt to construct the action of supermembranes on Calabi-Yau manifolds. In section 2, we begin with the preliminaries. The field decompositions on Calabi-Yau 3-folds are used to write down the supersymmetric invariant one-forms $\Pi$. However, these forms will not be invariant under supersymmetry transformations as soon as we lift them on Calabi-Yau manifolds. We then show that the supersymmetry transformations on a Calabi-Yau 3-fold can be deformed to look like the BRST transformations. This fact is then used to construct the supersymmetric DBI action of supermembranes.

## 2 Preliminaries and the construction of the action

Consider a Calabi-Yau 3-fold $M$ which has the holonomy group $SU(3)$. Take $\theta$ to be the singlet spinor on $M$ with the negative chirality. Then, as in [7, 6], we can use the following Fierz identities

$$\theta\theta^\dagger + \frac{1}{2}\gamma^{\bar{\alpha}}\theta^*\theta^t\gamma_{\bar{\alpha}} = \frac{1}{2}(1-\gamma_7) \tag{1}$$

$$\theta^*\theta^t + \frac{1}{2}\gamma^{\alpha}\theta\theta^\dagger\gamma_{\alpha} = \frac{1}{2}(1+\gamma_7) \tag{2}$$

to decompose a complex spinor $\Psi = \Psi_L + \Psi_R$ on $M$. Here we have used $\alpha, \beta, \gamma, \ldots$ to represent the complex tangent indices on $M$. This results in

$$\begin{aligned} \Psi_L &= \theta\psi + \frac{1}{2}\gamma^{\bar{\alpha}}\theta^*\psi_{\bar{\alpha}} \Rightarrow {\Psi^\dagger}_L = \bar{\psi}\theta^\dagger + \frac{1}{2}\bar{\psi}_\alpha\theta^t\gamma^\alpha \\ \Psi_R &= \theta^*\chi + \frac{1}{2}\gamma^{\alpha}\theta\chi_\alpha \Rightarrow {\Psi^\dagger}_R = \bar{\chi}\theta^t + \frac{1}{2}\bar{\chi}_{\bar{\alpha}}\theta^\dagger\gamma^{\bar{\alpha}}\,, \end{aligned} \tag{3}$$

where we have defined

$$\begin{aligned} \psi = \theta^\dagger\Psi_L \quad &, \quad \psi_{\bar{\alpha}} = \theta^t\gamma_{\bar{\alpha}}\Psi_L \\ \chi = \theta^t\Psi_R \quad &, \quad \chi_\alpha = \theta^\dagger\gamma_\alpha\Psi_R\,. \end{aligned}$$

As $\theta$ is a singlet, we have chosen $\gamma_\alpha\theta = 0$. The covariantly constant forms that can be constructed from $\theta$ are the Kähler 2-form $k_{\alpha\bar{\beta}} = i\theta^\dagger\gamma_{\alpha\bar{\beta}}\theta$, and the holomorphic 3-form $C_{\alpha\beta\gamma} = \theta^\dagger\gamma_{\alpha\beta\gamma}\theta^*$.

As mentioned in the introduction, the supermembrane action can exist in seven dimensions. Therefore we first write the action on flat $\mathbf{R}^7$ spacetime [1] using the above decomposition of fields, then we lift it on to $M\times\mathbf{R}^1$, where $\mathbf{R}^1$ represent the time direction.

Let us first recall the construction of the action on flat spacetime [1]. Let $m, n, \ldots = 0, 1, 2, \ldots, 6$ represent the tangent indices on the whole manifold, and $\mu, \nu, \ldots = 1, 2, \ldots, 6$ indicate the indices on $M$. $i, j, k, \ldots = 1, 2, 3$ will be the tangent indices on the worldvolume of the brane. As for the gamma matrices we take

$$\gamma_6 = \gamma_0\gamma_1\cdots\gamma_5\ , \qquad \gamma_7 = i\gamma_0\,,$$

where $\gamma_\mu$'s are hermitian and $\gamma_0^\dagger = -\gamma_0$. Define

$$\Pi^m = dX^m - i\bar{\Psi}\gamma^m d\Psi + id\bar{\Psi}\gamma^m\Psi \,,$$

where $\bar{\Psi} = \Psi^\dagger\gamma_0$. $\Pi^m$ is invariant under the following rigid supersymmetry transformations

$$\delta X^m = i\bar{\varepsilon}\gamma^m\Psi - i\bar{\Psi}\gamma^m\varepsilon \quad , \quad \delta\Psi = \varepsilon \,. \tag{4}$$

Here $\varepsilon = \varepsilon_L + \varepsilon_R$ is a constant complex six-dimensional spinor. On $M$, $\varepsilon$ decomposes just like $\Psi$, however, as we are interested in the scalar part of the transformations, we drop the triplet part of the $\varepsilon$;

$$\varepsilon_L = \theta\epsilon \quad , \qquad \varepsilon_R = \theta^*\eta \,. \tag{5}$$

Using the field decompositions in (3) and (5), the supersymmetry transformations (4) read (with $\phi \equiv X^0$)

$$\begin{aligned}
&\delta X^\alpha = \bar{\eta}\psi^\alpha + \epsilon\bar{\chi}^\alpha \\
&\delta X^{\bar{\alpha}} = -\eta\bar{\psi}^{\bar{\alpha}} - \bar{\epsilon}\chi^{\bar{\alpha}} \\
&\delta\phi = i\bar{\epsilon}\psi + i\epsilon\bar{\psi} + i\bar{\eta}\chi + i\eta\bar{\chi} \\
&\delta\psi = \epsilon \,, \qquad \delta\chi = \eta \\
&\delta\psi^\alpha = 0 \,, \qquad \delta\chi^{\bar{\alpha}} = 0 \,.
\end{aligned} \tag{6}$$

For the sake of simplicity, in the following, we set $\eta = 0$ in the above transformations and work with just one of the supersymmetries survived on $M$.

The components of $\Pi$, on the other hand, are

$$\begin{aligned}
\Pi^\alpha &= dX^\alpha - \bar{\chi}D\psi^\alpha - \psi^\alpha d\bar{\chi} - \psi D\bar{\chi}^\alpha - \bar{\chi}^\alpha d\psi + \frac{1}{4}C^{\alpha\beta\gamma}(\bar{\psi}_\beta D\chi_\gamma - \chi_\beta D\bar{\psi}_\gamma) \\
\Pi^{\bar{\alpha}} &= dX^{\bar{\alpha}} + \chi D\bar{\psi}^{\bar{\alpha}} + \bar{\psi}^{\bar{\alpha}} d\chi + \bar{\psi}D\chi^{\bar{\alpha}} + \chi^{\bar{\alpha}} d\bar{\psi} - \frac{1}{4}C^{\bar{\alpha}\bar{\beta}\bar{\gamma}}(\psi_{\bar{\beta}} D\bar{\chi}_{\bar{\gamma}} - \bar{\chi}_{\bar{\beta}} D\psi_{\bar{\gamma}}) \\
\Pi^0 &= d\phi - i\bar{\psi}d\psi - i\bar{\chi}d\chi - \frac{i}{2}g_{\alpha\bar{\beta}}\bar{\psi}^{\bar{\beta}}D\psi^\alpha - \frac{i}{2}g_{\alpha\bar{\beta}}\bar{\chi}^\alpha D\chi^{\bar{\beta}} \\
&\quad -i\psi d\bar{\psi} - i\chi d\bar{\chi} - \frac{i}{2}g_{\alpha\bar{\beta}}\psi^\beta D\bar{\psi}^{\bar{\alpha}} - \frac{i}{2}g_{\alpha\bar{\beta}}\chi^{\bar{\alpha}} D\bar{\chi}^\beta \,,
\end{aligned}$$

where $D\psi^\alpha = d\psi^\alpha + dX^\beta\Gamma^\alpha_{\beta\gamma}\psi^\gamma$. Further define

$$M_{ij} = \delta_{mn}\Pi^m_i\Pi^n_j \,. \tag{7}$$

Now as $\Pi$'s are invariant under the SUSY transformations on flat spacetime, the action

$$S_{DBI} = -T\int d^3\sigma\sqrt{\det M_{ij}} \,, \tag{8}$$

is also trivially invariant.

On a Calabi-Yau 3-fold, however, neither $\Pi$ nor the metric is invariant under the SUSY transformations. Therefore $M_{ij} = g_{mn}\Pi^m_i\Pi^n_j$ will not be invariant;

$$\delta(g_{mn}\Pi^m_i\Pi^n_j) = \Delta(g_{mn}\Pi^m_i\Pi^n_j) = g_{mn}(\Delta\Pi^m_i)\Pi^n_j + g_{mn}\Pi^m_i\Delta\Pi^n_j \,, \tag{9}$$

where the covariant variation is defined by

$$\Delta\Pi^\alpha = \delta\Pi^\alpha + \delta X^\rho \Gamma^\alpha_{\rho\sigma}\psi^\sigma \,,$$

and the last equality in (9) follows as the metric $g_{mn}$ is covariantly constant.

To get a supersymmetric action on $M \times \mathbf{R}^1$, firstly we note that on *flat* spacetimes the transformations (6) square to zero (BRST-like) when acting on any field except $\phi$. Secondly if we define the operator $\tilde{Q}$ by

$$\tilde{Q} = Q + \bar{Q} \quad , \quad \delta = i\epsilon Q + i\bar{\epsilon}\bar{Q} \,,$$

we can see that $M_{ij}$ in the action can be written as a BRST-exact term;

$$M_{ij} = \delta_{mn}\Pi^m_i\Pi^n_j = \{\tilde{Q} \;,\;\; \frac{i}{2}(\psi + \bar{\psi})\delta_{mn}\Pi^m_i\Pi^n_j\} \,. \tag{10}$$

This follows as $\{\tilde{Q} \;,\; \frac{i}{2}(\psi + \bar{\psi})\} = 1$. If we could maintain the BRST property of $\tilde{Q}$ on $M$, with a choice of $M_{ij}$ as in (10), it would be straightforward to construct the supersymmetric action on a Calabi-Yau 3-fold. All we need to do is to compute the term

$$\{\tilde{Q} \;,\; \frac{i}{2}(\psi + \bar{\psi})g_{mn}\Pi^m_i\Pi^n_j\} \tag{11}$$

to get $M_{ij}$ on $M \times \mathbf{R}^1$. The invariance of the action under $\tilde{Q}$ then follows as $\tilde{Q}^2 = 0$;

$$\begin{aligned} \{\tilde{Q}\,,\, S_{DBI}\} &= -\frac{T}{2}\int d^3\sigma \sqrt{\det M_{ij}}\, M^{ij}\{\tilde{Q}\,,\, M_{ij}\} \\ &= -\frac{T}{2}\int d^3\sigma \sqrt{\det M_{ij}}\, M^{ij}\{\tilde{Q}\,,\{\tilde{Q}\,,\;\; \frac{i}{2}(\psi + \bar{\psi})g_{mn}\Pi^m_i\Pi^n_j\}\} = 0 \,. \end{aligned} \tag{12}$$

In the following, we will see how this method works.

First of all, on $M \times \mathbf{R}^1$ we need to covariantize the transformations (6). The only transformation which needs to change is that of $\psi^\alpha$. So we write

$$\delta\psi^\alpha = -\delta X^\rho \Gamma^\alpha_{\rho\sigma}\psi^\sigma \,.$$

With this change, however, $\delta$ does not square to zero anymore when acting on $\psi^\alpha$. To maintain this property of $\delta$, we add another term proportional to the Riemann tensor as follows

$$\delta\psi^\alpha = -\epsilon\bar{\chi}^\rho\Gamma^\alpha_{\rho\sigma}\psi^\sigma - i\epsilon\bar{\psi}\bar{\chi}^\beta\chi^{\bar{\rho}}\psi^\gamma R^\alpha{}_{\beta\bar{\rho}\gamma} \,,$$

or

$$\Delta\psi^\alpha = -i\epsilon\bar{\psi}\bar{\chi}^\beta\chi^{\bar{\rho}}\psi^\gamma R^\alpha{}_{\beta\bar{\rho}\gamma} \,.$$

Noting that on Kähler manifolds $R^\alpha{}_{\beta\rho\gamma} = 0$ and $R^\alpha{}_{\beta\bar{\rho}\gamma} = R^\alpha{}_{\gamma\bar{\rho}\beta}$, it is easy to check that with this change $\delta_{\epsilon_1}\delta_{\epsilon_2}\psi^\alpha = 0$.

Taking $M_{ij}$ as in (11) we need to work out the variation of $\Pi$'s. Since the holomorphic 3-form $C^{\alpha\beta\gamma}$ is covariantly constant, we obtain

$$\Delta\Pi_i^\alpha = -\bar\chi\Delta(D_i\psi^\alpha) + \partial_i\bar\chi\Delta\psi^\alpha - \psi\Delta(D_i\bar\chi^\alpha) + \frac{1}{4}C^\alpha_{\ \bar\beta\bar\gamma}\left(\Delta\bar\psi^{\bar\beta}D_i\chi^{\bar\gamma} + \bar\psi^{\bar\beta}\Delta(D_i\chi^{\bar\gamma}) - \chi^{\bar\beta}\Delta(D_i\bar\psi^{\bar\gamma})\right)$$
$$\Delta\Pi_i^0 = -\frac{i}{2}g_{\alpha\bar\beta}(\Delta\bar\psi^{\bar\beta}D_i\psi^\alpha + \bar\psi^{\bar\beta}\Delta(D_i\psi^\alpha) + \bar\chi^\alpha\Delta(D_i\chi^{\bar\beta})) + \text{h.c.}\,,$$

where

$$\begin{aligned}\Delta(D_i\psi^\alpha) &= -(\epsilon\bar\chi^\beta\psi^\sigma X_i^{\bar\rho} + \bar\epsilon\chi^{\bar\rho}\psi^\sigma X_i^\beta)R^\alpha_{\ \beta\bar\rho\sigma} - i\epsilon D_i(\bar\psi\bar\chi^\beta\chi^{\bar\rho}\psi^\gamma R^\alpha_{\ \beta\bar\rho\gamma})\\ \Delta(D_i\bar\chi^\alpha) &= -\bar\epsilon X_i^\beta\chi^{\bar\rho}\bar\chi^\gamma R^\alpha_{\ \beta\bar\rho\gamma}\,.\end{aligned} \tag{13}$$

So finally we find the following supersymmetric action for membranes on $M \times \mathbf{R}^1$

$$S_{DBI} = -T\int d^3\sigma\sqrt{\det M_{ij}}\,, \tag{14}$$

with

$$\begin{aligned}M_{ij} &= g_{mn}\Pi_i^m\Pi_j^n\\ &- \frac{i}{2}(\psi+\bar\psi)g_{\alpha\bar\beta}\Pi^{\bar\beta}_{(j}\Big[(i\bar\chi\bar\chi^\beta\psi^\sigma X^{\bar\rho}_{i)} + i\bar\chi\chi^{\bar\rho}\psi^\sigma X^\beta_{i)}\\ &+ \partial_{i)}\bar\chi\bar\psi\bar\chi^\beta\chi^{\bar\rho}\psi^\sigma + i\psi\chi^{\bar\rho}\bar\chi^\sigma X^\beta_{i)})R^\alpha_{\ \beta\bar\rho\sigma} - \bar\chi D_{i)}(\bar\psi\bar\chi^\beta\chi^{\bar\rho}\psi^\gamma R^\alpha_{\ \beta\bar\rho\gamma})\\ &+ \frac{1}{4}C^\alpha_{\ \bar\beta\bar\gamma}\Big((\psi\chi^{\bar\sigma}\bar\chi^\rho\bar\psi^{\bar\eta}D_{i)}\chi^{\bar\gamma} + i\bar\psi^{\bar\gamma}\bar\chi^\rho\chi^{\bar\eta}X^{\bar\sigma}_{i)}\\ &- i\chi^{\bar\gamma}\chi^{\bar\sigma}\bar\psi^{\bar\eta}X^\rho_{i)} - i\chi^{\bar\gamma}\bar\chi^\rho\bar\psi^{\bar\eta}X^{\bar\sigma}_{i)})R^{\bar\beta}_{\ \bar\sigma\rho\bar\eta} + \chi^{\bar\beta}D_{i)}(\psi\chi^{\bar\sigma}\bar\chi^\rho\bar\psi^{\bar\eta}R^{\bar\gamma}_{\ \bar\sigma\rho\bar\eta})\Big)\Big]\\ &+ \frac{i}{4}(\psi+\bar\psi)g_{\alpha\bar\beta}\Pi^0_{(j}\Big[-i\psi\chi^{\bar\sigma}\bar\chi^\rho\bar\psi^{\bar\gamma}D_{i)}\psi^\alpha R^{\bar\beta}_{\ \bar\sigma\rho\bar\gamma}\\ &- \bar\psi^{\bar\beta}(\bar\chi^\sigma\psi^\gamma X^{\bar\rho}_{i)} + \chi^{\bar\rho}\psi^\gamma X^\sigma_{i)})R^\alpha_{\ \sigma\bar\rho\gamma} - i\bar\psi^{\bar\beta}D_{i)}(\bar\psi\bar\chi^\beta\chi^{\bar\rho}\psi^\gamma R^\alpha_{\ \beta\bar\rho\gamma})\Big] + \text{h.c.}\,.\end{aligned}$$

As mentioned earlier, since $\delta_{\epsilon_2}\delta_{\epsilon_1}$ acting on any field[1] gives zero it follows that $\tilde{Q}^2 = 0$, which ensures the invariance of the action under $\tilde{Q}$.

Apart from the DBI action which was constructed above, supermembranes action on flat spacetime has another part, the Wess-Zumino term. One way to obtain the WZ term is to vary the DBI action with respect to the $\kappa$-symmetry transformations. One then looks for a supersymmetric (up to a total derivative) 3-form on the worldvolume such that its variation under the $\kappa$-transformations cancels the variation of DBI action. The whole action $S = S_{DBI} + S_{WZ}$ is then invariant under both the supersymmetry and $\kappa$-symmetry transformations. We hope to return to these issues in future works.

[1] $\delta^2$ acting on $\phi$ does not give zero, but this does not cause any harm as this field appears in $\Pi^0$ as $d\phi$ and $\delta^2$ acting on $\Pi^0$ still gives zero.

## References


[1] P.K. Townsend, *Three lectures on supermembranes*, in Superstrings '88, eds. M. Green, M. Grisaru, R. Iengo, E. Sezgin and A. Strominger, World Scientific 1989.

[2] P. Dawson, *D1 and D5-brane actions in* $AdS_m \times S^n$, hep-th/0002030.

[3] M. Bershadsky, V. Sadov, and C. Vafa, *D-branes and topological field theory*, Nucl. Phys. **B463** (1996), 420, hep-th/9511222.

[4] K. Becker, M. Becker and A. Strominger, *Fivebranes, Membranes and Non-Perturbative String Theory*, Nucl.Phys. **B456** (1995) 130, hep-th/9507158.

[5] E. Witten, *Bound states of strings and p-branes*, Nucl. Phys. **B460** (1996), 335, hep-th/9510135

[6] A. Imaanpur, *A 3d topological sigma model and D-branes*, JHEP 9909(1999)010, hep-th/9906131.

[7] J.M. Figueroa-O'Farrill, A. Imaanpur and J. McCarthy, *Supersymmetry and gauge theory on Calabi-Yau 3-folds*, Phys. Lett. **B419** (1998) 167, hep-th/9709178.